Change in the mean in the domain of attraction of the normal law via Darling-Erdős theorems

Miklós Csörgő*
School of Mathematics and Statistics
Carleton University, 1125 Colonel By Drive, Ottawa, ON K1S 5B6, Canada
mcsorgo@math.carleton.ca

Zhishui Hu†
Department of Statistics and Finance, School of Management
University of Science and Technology of China, Hefei, Anhui 230026, China
huzs@ustc.edu.cn

Abstract. This paper studies the problem of testing the null assumption of no-change in the mean of chronologically ordered independent observations on a random variable X *versus* the at most one change in the mean alternative hypothesis. The approach taken is via a Darling-Erdős type self-normalized maximal deviation between sample means before and sample means after possible times of a change in the expected values of the observations of a random sample. Asymptotically, the thus formulated maximal deviations are shown to have a standard Gumbel distribution under the null assumption of no change in the mean. A first such result is proved under the condition that $EX^2 \log\log(|X|+1) < \infty$, while in the case of a second one, X is assumed to be in a specific class of the domain of attraction of the normal law, possibly with infinite variance.

Key Words: Change in the mean, domain of attraction of the normal law, Darling-Erdős theorems, Gumbel distribution, weighted metrics, Brownian bridge.

AMS 2000 Subject Classification: Primary 60F05; secondary 62G10.

*Research supported by an NSERC Canada Discovery Grant at Carleton University.
†Partially supported by NSFC(No.10801122) and RFDP(No.200803581009), and by an NSERC Canada Discovery Grant of M. Csörgő at Carleton University.

1 Introduction and main results

Let $X, X_1, X_2, \cdots$ be non-degenerate independent identically distributed (i.i.d.) real-valued random variables (r.v.'s) with a finite mean $EX = \mu$. We are interested in testing the null assumption

$$H_0: \ X_1, X_2, \cdots X_n \text{ is a random sample on } X \text{ with a finite mean } EX = \mu$$

versus the "at most one change in the mean" (AMOC) alternative hypothesis

$$\begin{aligned} H_A: \ &\text{there is an integer } k^*, 1 \le k^* < n \text{ such that} \\ &EX_1 = \cdots = EX_{k^*} \ne EX_{k^*+1} = \cdots = EX_n. \end{aligned}$$

The hypothesized time k^* of at most one change in the mean is usually unknown. Hence, given *chronologically ordered* independent observables $X_1, X_2, \cdots, X_n, n \ge 1$, in order to test H_0 versus H_A, from a non-parametric point of view it appears to be reasonable to compare the sample mean $(X_1 + \cdots + X_k)/k =: S_k/k$ at any time $1 \le k < n$ to the sample mean $(X_{k+1} + \cdots + X_n)/(n-k) =: (S_n - S_k)/(n-k)$ after time $1 \le k < n$ via functionals in k of the family of the standardized statistics

$$\begin{aligned} \Gamma_n(k) &:= \Big(n\frac{k}{n}\Big(1-\frac{k}{n}\Big)\Big)^{1/2}\Big(\frac{S_k}{k} - \frac{S_n - S_k}{n-k}\Big) \\ &= \frac{1}{(\frac{k}{n}(1-\frac{k}{n}))^{1/2}}\Big(\frac{S_k}{n^{1/2}} - \frac{k}{n}\frac{S_n}{n^{1/2}}\Big), \quad 1 \le k < n. \end{aligned} \tag{1.1}$$

For instance, one would want to reject H_0 in favor of H_A for large observed values of

$$\Gamma_n := \max_{1 \le k < n} |\Gamma_n(k)|. \tag{1.2}$$

On the other hand, when assuming for example that the independent observables $X_1, \cdots, X_n, n \ge 1$, are $N(\mu, \sigma^2)$ random variables, then we find ourselves modeling and testing for a parametric shift in the mean AMOC problem. It is, however, easy to check that, when the variance σ^2 is known, then

$$-2 \log \Lambda_k = \frac{1}{\sigma^2}(\Gamma_n(k))^2, \tag{1.3}$$

where Λ_k is the *likelihood ratio statistic* if the change in the mean occurs at $k^* = k$. Hence, the *maximally selected likelihood ratio statistic* $\max_{1 \le k < n}(-2 \log \Lambda_k)$ will be large if and only if Γ_n of (1.2) is large. A similar conclusion holds true if the variance σ^2 is an unknown but constant nuisance parameter (cf. Gombay and Horváth (1994, 1996a,b), and Csörgő and Horváth (1997) [Section 1.4], and references therein). Namely in this case the maximally selected likelihood ratio statistic $\max_{1 \le k < n}(-2 \log \Lambda_k)$ will be large if and only if

$$\hat{\Gamma}_k := \max_{1 \le k < n} \frac{1}{\hat{\sigma}_{k,n}} |\Gamma_n(k)| \tag{1.4}$$

is large, where

$$\hat{\sigma}_{k,n}^2 := \frac{1}{n}\Big\{\sum_{1\le i\le k}\Big(X_i - \frac{S_k}{k}\Big)^2 + \sum_{k<i\le n}\Big(X_i - \frac{S_n - S_k}{n-k}\Big)^2\Big\}. \tag{1.5}$$

These conclusions, and further examples as well in Csörgő and Horváth (1988) [Section 2], and in Csörgő and Horváth (1997) [Section 1.4] that are based on Gombay and Horváth (1994, 1996a,b), show that under the null hypothesis H_0 a large number of parametric and nonparametric modeling of AMOC problems result in the same test statistic, namely that of (1.2), or its variant in (1.4). Consequently, if the underlying distribution is not known, the just mentioned test statistics should continue to work just as well when testing for H_0 versus H_A as above. Furthermore, Brodsky and Darkhovsky (1993) argue quite convincingly in their Section 1.2 that detecting changes in the mean (mathematical expectation) of a random sequence constitutes one basic situation to which other changes in distribution can be conveniently reduced. Thus Γ_n and $\hat{\Gamma}_n$ gain a somewhat focal role in change-point analysis in general as well. Studying the asymptotic behavior of these statistics is clearly of interest.

Let $S_0 = 0$, and for $n \ge 1$ define the sequence of tied-down partial sums processes

$$Z_n(t) := \begin{cases} (S_{[(n+1)t]} - [(n+1)t]S_n/n)/n^{1/2}, & 0 \le t < 1,\\ 0, & t = 1. \end{cases} \tag{1.6}$$

In view of (1.1), we are interested in exploring the asymptotic behavior of the standardized sequence of stochastic processes

$$\Big\{\frac{1}{(t(1-t))^{1/2}} Z_n(t), 0 \le t < 1\Big\}.$$

We first note that

$$\sup_{0<t<1} \frac{1}{\sigma}|Z_n(t)|/(t(1-t))^{1/2}$$

and, naturally, also the standardized statistics Γ_n and $\hat{\Gamma}_n$ (cf. (1.2) and (1.4)) converge in distribution to ∞ as $n \to \infty$ even if the null assumption of no change in the mean is true. Hence, in order to secure nondegenerate limiting behavior under H_0, we seek appropriate renormalizations.

For example, it is proved in Csörgő, Szyszkowicz and Wang (2004) (cf. Corollary 5.2 in there) that, on *assuming X to be in the domain of attraction of the normal law (DAN), possibly with infinite variance*, then, as $n \to \infty$,

$$\sup_{0<t<1} \frac{1}{\hat{\sigma}_{[nt+1],n}}|Z_n(t)|/q(t) \xrightarrow{d} \sup_{0<t<1} |B(t)|/q(t), \tag{1.7}$$

where $\{B(t), 0 \le t \le 1\}$ is a Brownian bridge, $\hat{\sigma}_{k,n}$, $1 \le k \le n-1$ is as in (1.5), $\hat{\sigma}_{n,n}^2 := \frac{1}{n}\sum_{1\le i\le n}(X_i - \frac{S_n}{n})^2$,

$$q(t) := \begin{cases} (t\log\log(t^{-1}))^{1/2}, & t \in (0, 1/2],\\ ((1-t)\log\log((1-t)^{-1}))^{1/2}, & t \in [1/2, 1), \end{cases}$$

and $\log x := \log(\max\{e, x\})$.

Large values of the statistics in (1.7) indicate evidence against H_0. The weight function $q(\cdot)$ is to emphasize changes that may have recurred near 0 and n. We note in passing that the result in (1.7) cannot be deduced via first proving a "corresponding" weak invariance principle on $D[0,1]$ (cf. Csörgő *et al.* (2004), Remark 5.2, as well as Corollaries 2 and 4 of Csörgő *et al.* (2008a) and their extension (46) in Theorem 4 of Csörgő *et al.* (2008b)). The applicability of (1.7) is much enhanced by Orasch and Pouliot (2004), tabulating functionals in weighted sup-norm.

An alternative way of studying change in the mean is via Darling-Erdős type theorems. For example (cf. Theorems 2.1.2, A.4.2 and Corollary 2.1.2 in Csörgő and Horváth (1997)), *under* H_0 *with* $EX^2 \log\log(|X|+1) < \infty$*, we have*

$$\lim_{n\to\infty} P\Big(a(n) \max_{1\le k<n} \frac{1}{\hat{\sigma}_{k,n}} \Big(\frac{n^2}{k(n-k)}\Big)^{1/2} Z_n\Big(\frac{k}{n+1}\Big) \le t + b(n)\Big) = \exp(-e^{-t}), \quad t \in \mathbb{R}, \tag{1.8}$$

where

$$a(n) := (2\log\log n)^{1/2} \quad \textit{and} \quad b(n) := 2\log\log n + \frac{1}{2}\log\log\log n - \frac{1}{2}\log \pi. \tag{1.9}$$

In view of (1.7), the aim of this paper is to explore the possibility of extending the result of (1.8) to versions of $Z_n(\frac{k}{n+1})$ under H_0 with $X \in$ DAN, for the sake of having an alternative approach to the sup-norm procedure of (1.7) for studying the problem of a change in the mean in DAN, possibly with $EX^2 = \infty$.

Define the family of statistics

$$T_{k,n} = \frac{\frac{S_k}{k} - \frac{S_n - S_k}{n-k}}{\sqrt{\frac{\sum_{i=1}^{k}(X_i - S_k/k)^2}{k(k-1)} + \frac{\sum_{i=k+1}^{n}(X_i - (S_n - S_k)/(n-k))^2}{(n-k)(n-k-1)}}}, \quad 2 \le k \le n-2. \tag{1.10}$$

We note in passing that, on writing

$$\tilde{\sigma}_{k,n}^2 := \frac{\sum_{1\le i\le k}\Big(X_i - \frac{S_k}{k}\Big)^2}{k(k-1)} + \frac{\sum_{k<i\le n}\Big(X_i - \frac{S_n - S_k}{n-k}\Big)^2}{(n-k)(n-k-1)}, \quad 2 \le k \le n-2, \tag{1.11}$$

we get

$$T_{k,n} = \frac{1}{\tilde{\sigma}_{k,n}} \Big(\frac{n}{k(n-k)}\Big)^{1/2} \Big(\frac{n^2}{k(n-k)}\Big)^{1/2} Z_n\Big(\frac{k}{n+1}\Big), \quad 2 \le k \le n-2. \tag{1.12}$$

We note also that $(k(n-k)/n)\tilde{\sigma}_{k,n}^2$ is an unbiased estimator of σ^2 when $EX^2 < \infty$.

Our first result is to say that, under the same moment condition for X, the self-normalized statistics $\max_{2\le k\le n-2} T_{k,n}$ behaves like $\max_{1\le k<n} \frac{1}{\hat{\sigma}_{k,n}}(\frac{n^2}{k(n-k)})^{1/2} Z_n(\frac{k}{n+1})$ does asymptotically (cf. our Theorem 1.1 and (1.8)). Our main result, Theorem 1.2, however concludes the same asymptotic behavior for $\max_{1\le k<n} T_{k,n}$ for $X \in$ DAN with possibly infinite variance.

Theorem 1.1. *Assume that H_0 holds and*

$$EX^2 \log\log(|X|+1) < \infty. \tag{1.13}$$

Then

$$\lim_{n\to\infty} P\Big(a(n) \max_{2\le k\le n-2} T_{k,n} \le t + b(n)\Big) = \exp(-e^{-t}), \quad t \in \mathbb{R}.$$

Write $l(x) := E(X-\mu)^2 I(|X-\mu| \le x)$. Assume that X belongs to the domain of attraction of the normal law. Then $l(x)$ is a slowly varying function as $x \to \infty$. Consequently, there exists some $a > 1$ such that for any $x > a$ (see, for example, Galambos and Seneta (1973)),

$$\ell(x) = \exp\Big\{c(x) + \int_a^x \frac{\varepsilon(t)}{t} dt\Big\}, \tag{1.14}$$

where $c(x) \to c(|c| < \infty)$ as $x \to \infty$ and $\varepsilon(t) \to 0$ as $t \to \infty$.

Theorem 1.2. *Assume that H_0 holds and $l(x)$ is a slowly varying function at ∞ that, in terms of the representation (1.14), satisfies the additional conditions $c(x) \equiv c$ and $\varepsilon(t) \le C_0/\log t$ for some $C_0 > 0$, i.e., $X \in DAN$, possibly with infinite variance, under the latter specific conditions on $l(x)$. Then, for all $t \in \mathbb{R}$,*

$$\lim_{n\to\infty} P\Big(a(n) \max_{2\le k\le n-2} T_{k,n} \le t + b(n)\Big) = \exp(-e^{-t}).$$

Remark 1. The additional conditions in Theorem 1.2 are satisfied by a large class of slowly varying functions, such as $l(x) = (\log\log x)^\alpha$ and $l(x) = (\log x)^\alpha$, for example, for some $0 < \alpha < \infty$.

Remark 2. Csörgő, Szyszkowicz and Wang (2003) obtained the follwoing Darling-Erdős theorem for self-normalized sums: *suppose that H_0 holds with $EX = 0$ and $l(x)$ is a slowly varying function at ∞, satisfying*

$$l(x^2) \le Cl(x) \quad \textit{for some} \quad C > 0. \tag{1.15}$$

Then, for every $t \in \mathbb{R}$,

$$\lim_{n\to\infty} P\Big(a(n) \max_{1\le k\le n} S_k/V_k \le t + b(n)\Big) = \exp(-e^{-t}).$$

If $l(x)$ has the representation (1.14) with $c(x) \equiv c$ and $\varepsilon(t) \le C_0/\log t$ for some $C_0 > 0$, then

$$\frac{l(x^2)}{l(x)} = \exp\Big\{\int_x^{x^2} \frac{\varepsilon(t)}{t} dt\Big\} \le \exp\Big\{C_0 \int_x^{x^2} \frac{1}{t\log t} dt\Big\} = 2^{C_0}.$$

So, (1.15) holds under the additional smoothness conditions for $l(x)$ that are needed for results like Lemma 2.1, for example. On the other hand, if $\varepsilon(x) = (\log x)^{-\alpha}$ for some

$0<\alpha<1$, then $\lim_{x\to\infty} l(x^2)/l(x)=\infty$, i.e., (1.15) fails. Thus, the additional conditions on $l(x)$ in Theorem 1.2 that are sufficient for having (1.15), are seen to be not far from being also necessary.

Before proving Theorems 1.1 and 1.2, we pose the following question.

Question 1. In view of Theorems 1.1 and 1.2, one may like to know if the result of (1.8) could also hold true when replacing condition (1.13) by $X \in$ DAN, possibly with $EX^2=\infty$.

Question 2. In view of having Theorems 1.1 and 1.2, one would hope to have (1.7) in terms of $T_{k,n}$, i.e., when replacing $\frac{1}{\hat{\sigma}_{[nt+1],n}}$ by $\frac{1}{\tilde{\sigma}_{[nt+1],n}}(\frac{n}{[nt+1](n-[nt])})^{1/2}$ on the left hand side of (1.7), with $\tilde{\sigma}_{k,n},\ 1\le k\le n-1$ defined as in (1.11) and $\tilde{\sigma}^2_{n,n}:=\frac{1}{n^2}\sum_{1\le i\le n}(X_i-\frac{S_n}{n})^2$.

As to these questions, it is clear from the respective proofs of (1.8) (cf. Corollary 2.1.2 in Csörgő and Horváth (1997)) and Theorem 1.1 that, under the condition (1.13), the two estimators $\hat{\sigma}^2_{k,n}$ and $(k(n-k)/n)\tilde{\sigma}^2_{k,n}$ of σ^2 are asymptotically equivalent. When $\mathrm{Var}(X)=\infty$, this does not appear to be true any more, i.e., when these "estimators" in hand are being used as self-normalizers. However, we could not resolve this problem as posed in the context of these two questions.

2 Proofs of Theorems 1.1 and 1.2

Without loss of generality, in this section we assume that $\mu=0$.

Proof of Theorem 1.1. Write $K_n=\exp\{\log^{1/3}n\}$. With $\tilde{\sigma}^2_{k,n}$ as in (1.11), in view of (1.12), at first, we prove that, as $n\to\infty$,

$$\max_{K_n<k<n-K_n}\Big|\frac{k(n-k)}{n}\tilde{\sigma}^2_{k,n}-\sigma^2\Big| = o_P((\log\log n)^{-1}). \tag{2.1}$$

Write $\tilde{b}_n=n/\log\log n$. Then $\tilde{b}_n/n\downarrow$ and $\tilde{b}_n^2\sum_{i=n}^{\infty}\tilde{b}_i^{-2}=O(n)$. Noting that, for sufficiently large n, we have

$$\begin{aligned} P\Big(|X^2-\sigma^2|>\tilde{b}_n\Big) &\le P\Big(|X^2-\sigma^2|\log\log(|X^2-\sigma^2|+1)>\tilde{b}_n\log\log\tilde{b}_n\Big)\\ &\le P\Big(|X^2-\sigma^2|\log\log(|X^2-\sigma^2|+1)>(1/2)n\Big), \end{aligned}$$

and $E|X^2-\sigma^2|\log\log(|X^2-\sigma^2|+1)<\infty$ (by the assumption $EX^2\log\log(|X|+1)<\infty$), we conclude

$$\sum_{n=1}^{\infty}P\Big(|X^2-\sigma^2|>\frac{n}{\log\log n}\Big)<\infty.$$

By Theorem 3 in Chow and Teicher (1978, page 126), we get

$$\sum_{i=1}^{k}(X_i^2-\sigma^2)=o(k(\log\log k)^{-1})\quad a.s.\ \text{ as } k\to\infty.$$

Hence, by the classical Hartman-Wintner LIL, as $k \to \infty$, we have

$$\sum_{i=1}^{k}(X_i - S_k/k)^2 - k\sigma^2 = \sum_{i=1}^{k}(X_i^2 - k\sigma^2) - S_k^2/k = o(k(\log\log k)^{-1}) \quad a.s.$$

Consequently,

$$\max_{K_n<k\le n}\Big|\frac{1}{k}\sum_{i=1}^{k}(X_i - S_k/k)^2 - \sigma^2\Big| = o_P((\log\log n)^{-1}),$$

and

$$\max_{1\le k<n-K_n}\Big|\frac{1}{n-k}\sum_{i=k+1}^{n}(X_i - (S_n - S_k)/(n-k))^2 - \sigma^2\Big| = o_P((\log\log n)^{-1}).$$

Hence (2.1) holds.

By Theorem 2.1.2 in Csörgő and Horváth (1997), we have

$$(2\log\log n)^{-1/2}\max_{1\le k<n}\Big(\frac{n}{k(n-k)}\Big)^{1/2}\Big|S_k - \frac{k}{n}S_n\Big| \xrightarrow{P} \sigma.$$

This, together with (2.1), implies

$$\begin{aligned}
&a(n)\Big|\max_{K_n<k<n-K_n} T_{k,n} - \frac{1}{\sigma}\max_{K_n<k<n-K_n}\Big(\frac{n}{k(n-k)}\Big)^{1/2}\Big(S_k - \frac{k}{n}S_n\Big)\Big| \\
&\le a(n)\max_{K_n<k<n-K_n}\Big(\frac{n}{k(n-k)}\Big)^{1/2}\Big|S_k - \frac{k}{n}S_n\Big|\Big|\Big(\frac{k(n-k)}{n}\tilde{\sigma}_k^2\Big)^{-1/2} - \sigma^{-1}\Big| \\
&= o_P(1)(\log\log n)^{-1/2}\max_{1\le k<n}\Big(\frac{n}{k(n-k)}\Big)^{1/2}\Big|S_k - \frac{k}{n}S_n\Big| \xrightarrow{P} 0.
\end{aligned}$$

Then from the proof of Theorem A.4.2. in Csörgő and Horváth (1997), for all $t \in \mathbb{R}$, it follows that

$$\lim_{n\to\infty} P\Big(a(n)\max_{K_n<k<n-K_n} T_{k,n} \le t + b(n)\Big) = \exp(-e^{-t}). \tag{2.2}$$

Similarly to the proof of (2.26) and (2.27) below, we get

$$a(n)\max_{2\le k\le K_n} T_{k,n} - b(n) \xrightarrow{P} -\infty, \tag{2.3}$$

and

$$a(n)\max_{n-K_n\le k\le n-2} T_{k,n} - b(n) \xrightarrow{P} -\infty. \tag{2.4}$$

Now Theorem 1.1 follows from (2.2)–(2.4). $\square$

We continue with establishing three auxiliary lemmas for the proof of Theorem 1.2.

As in Csörgő *et al.* (2003), we start with putting $b = \inf\{x \geq 1; l(x) > 0\}$ and

$$\eta_n = \inf\Big\{s : s \geq b+1, \frac{l(s)}{s^2} \leq \frac{(\log\log n)^4}{n}\Big\}.$$

Let

$$Z_j = X_j I(|X_j| > \eta_j), \quad Y_j = X_j I(|X_j| \leq \eta_j), \quad Y_j^* = Y_j - EY_j,$$
$$S_n^* = \sum_{j=1}^n Y_j^*, \qquad B_n^2 = \sum_{j=1}^n EY_j^{*2}, \qquad V_n^2 = \sum_{j=1}^n X_j^2.$$

Then, as $n \to \infty$, $\eta_n \to \infty$, $nl(\eta_n) = \eta_n^2(\log\log n)^4$ for every large enough n and $B_n^2 \sim nl(\eta_n)$. As in Csörgő *et al.* (2003), we may assume without loss of generality that

$$B_n^2 = nl(\eta_n) = \eta_n^2(\log\log n)^4 \quad \text{for all} \quad n \geq 1.$$

Let $\{\tilde{X}, \tilde{X}_1, \tilde{X}_2, \cdots\}$ be a sequence of i.i.d. random variables with $\tilde{X} \stackrel{d}{=} X$, independently of $\{X, X_1, X_2, \cdots\}$. We define $\tilde{S}_n$, $\tilde{Z}_j$, $\tilde{Y}_j$, $\tilde{Y}_j^*$, $\tilde{S}_n^*$ and $\tilde{V}_n$ similarly to S_n, Z_j, Y_j, Y_j^*, S_n^* and V_n. Define

$$S_{k,n} = \begin{cases} \frac{S_k}{k} - \frac{\tilde{S}_{n-[n/2]}+S_{[n/2]}-S_k}{n-k}, & \text{if } 1 \leq k \leq n/2; \\ \frac{S_{[n/2]}+\tilde{S}_{n-[n/2]}-\tilde{S}_{n-k}}{k} - \frac{\tilde{S}_{n-k}}{n-k}, & \text{if } n/2 < k < n, \end{cases}$$

$$S_{k,n}^* = \begin{cases} \frac{S_k^*}{k} - \frac{\tilde{S}^*_{n-[n/2]}+S^*_{[n/2]}-S^*_k}{n-k}, & \text{if } 1 \leq k \leq n/2; \\ \frac{S^*_{[n/2]}+\tilde{S}^*_{n-[n/2]}-\tilde{S}^*_{n-k}}{k} - \frac{\tilde{S}^*_{n-k}}{n-k}, & \text{if } n/2 < k < n, \end{cases}$$

$$B_{k,n}^2 = \begin{cases} \frac{B_k^2}{k^2} + \frac{B^2_{n-[n/2]}+B^2_{[n/2]}-B^2_k}{(n-k)^2}, & \text{if } 1 \leq k \leq n/2; \\ \frac{B^2_{[n/2]}+B^2_{n-[n/2]}-B^2_{n-k}}{k^2} + \frac{B^2_{n-k}}{(n-k)^2}, & \text{if } n/2 < k < n, \end{cases}$$

$$V_{k,n}^2 = \begin{cases} \frac{V_k^2}{k^2} - \frac{S_k^2}{k^3} + \frac{\tilde{V}^2_{n-[n/2]}+V^2_{[n/2]}-V^2_k}{(n-k)^2} - \frac{(\tilde{S}_{n-[n/2]}+S_{[n/2]}-S_k)^2}{(n-k)^3}, & \text{if } 1 \leq k \leq n/2; \\ \frac{V^2_{[n/2]}+\tilde{V}^2_{n-[n/2]}-\tilde{V}^2_{n-k}}{k^2} - \frac{(S_{[n/2]}+\tilde{S}_{n-[n/2]}-\tilde{S}_{n-k})^2}{k^3} + \frac{\tilde{V}^2_{n-k}}{(n-k)^2} - \frac{\tilde{S}^2_{n-k}}{(n-k)^3}, & \text{if } n/2 < k < n. \end{cases}$$

$$\bar{V}_{k,n}^2 = \begin{cases} \frac{V_k^2}{k(k-1)} - \frac{S_k^2}{k^2(k-1)} + \frac{\tilde{V}^2_{n-[n/2]}+V^2_{[n/2]}-V^2_k}{(n-k)(n-k-1)} - \frac{(\tilde{S}_{n-[n/2]}+S_{[n/2]}-S_k)^2}{(n-k)^2(n-k-1)}, & \text{if } 2 \leq k \leq n/2; \\ \frac{V^2_{[n/2]}+\tilde{V}^2_{n-[n/2]}-\tilde{V}^2_{n-k}}{k(k-1)} - \frac{(S_{[n/2]}+\tilde{S}_{n-[n/2]}-\tilde{S}_{n-k})^2}{k^2(k-1)} + \frac{\tilde{V}^2_{n-k}}{(n-k)(n-k-1)} - \frac{\tilde{S}^2_{n-k}}{(n-k)^2(n-k-1)}, & \text{if } n/2 < k \leq n-2. \end{cases}$$

Clearly, with $\{T_{k,n}, 2 \leq k \leq n-2\}$ as in (1.10), we have

$$\{T_{k,n}, 2 \leq k \leq n-2\} \stackrel{d}{=} \Big\{\frac{S_{k,n}}{\bar{V}_{k,n}}, 2 \leq k \leq n-2\Big\} \quad \text{for each} \quad n \geq 4,$$

where, and throughout, $\stackrel{d}{=}$ stands for equality in distribution.

Lemma 2.1. *As $n \to \infty$, we have*

$$\frac{l(\eta_n) - l(\eta_{n/(\log\log n)^5})}{l(\eta_n)} = o(1/\log\log n). \tag{2.5}$$

Proof. Since

$$\begin{aligned} 1 \geq \frac{l(\eta_{n/(\log\log n)^5})}{l(\eta_n)} &\geq \exp\Big\{ - C_0 \int_{\eta_{n/(\log\log n)^5}}^{\eta_n} \frac{1}{u \log u} du\Big\} \\ &\geq \exp\Big\{ - C_0 \frac{\eta_n}{\eta_{n/(\log\log n)^5} \log \eta_{n/(\log\log n)^5}}\Big\}, \end{aligned}$$

and η_n is a regularly varying function with index $1/2$, for any $\varepsilon > 0$, we have $\eta_n/\eta_{n/(\log\log n)^5} \leq (\log\log n)^{5/2+\varepsilon}$ for sufficiently large n, and $\log \eta_{n/(\log\log n)^5} \sim (1/2)\log n$ as $n \to \infty$. Hence

$$\frac{l(\eta_n) - l(\eta_{n/(\log\log n)^5})}{l(\eta_n)} = o(1/\log\log n). \quad \square$$

Lemma 2.2. *As $n \to \infty$, we have*

$$\frac{\sum_{j=1}^n (|Z_j| + E|Z_j|)}{B_n/\sqrt{\log\log n}} \xrightarrow{P} 0.$$

Proof. Let $\tau_j = \eta_j(\log\log j)^3$ and $Z_j^* = X_j I(\eta_j < |X_j| < \tau_j)$. From the proof of Lemma 2 in Csörgő *et al.* (2003), we have $P(Z_j \neq Z_j^*, i.o.) = 0$. Hence, by Chebyshev's inequality, in order to prove Lemma 2.2, we only need to prove that, as $n \to \infty$,

$$\sum_{j=1}^n E|Z_j^*| = o(B_n/\sqrt{\log\log n}), \tag{2.6}$$

$$\sum_{j=1}^n EZ_j^{*2} = o(B_n^2/\log\log n), \tag{2.7}$$

$$\sum_{j=1}^n E|X_j|I(|X_j| > \tau_j) = o(B_n/\sqrt{\log\log n}). \tag{2.8}$$

We only prove (2.6) and (2.8), for the proof of (2.7) is similar to that of (2.6). Since η_n is a regularly varying function with index $1/2$, we have that for sufficiently large n,

$$\eta_{n/(\log\log n)^{16}}(\log\log n)^3 \leq \eta_{n/(\log\log n)^9}.$$

Also, similarly, by the fact that $\sqrt{j}(\log\log j)^2/\sqrt{l(\eta_j)}$ is a regularly varying function with index $1/2$, we have that for sufficiently large n,

$$\max_{1\leq j\leq n/(\log\log n)^9} \frac{j}{\eta_j} = \max_{1\leq j\leq n/(\log\log n)^9} \frac{\sqrt{j}(\log\log j)^2}{\sqrt{l(\eta_j)}} \leq \frac{\sqrt{n}}{\sqrt{l(\eta_n)}(\log\log n)^2}.$$

Hence, by using the same method as that in the proof of Lemma 2.1, we have

$$\begin{aligned}\sum_{j=1}^{n} E|Z_j^*| &\leq \sum_{j=1}^{n/(\log\log n)^{16}} E|X_1|I(\eta_i < |X_1| < \eta_{n/(\log\log n)^9}) \\ &\quad + nE|X_1|I(\eta_{n/(\log\log n)^{16}} < |X_1| < \eta_n(\log\log n)^3) \\ &\leq \sum_{j=1}^{n/(\log\log n)^9} jE|X_1|I(\eta_j < |X_1| < \eta_{j+1}) \\ &\quad + \frac{n(l(\eta_n(\log\log n)^3) - l(\eta_{n/(\log\log n)^{16}}))}{\eta_{n/(\log\log n)^{16}}} \\ &= o(B_n/(\log\log n)), \quad n \to \infty.\end{aligned}$$

Thus (2.6) is proved.

Next, we prove (2.8). By the fact that $E|X|I(|X| \geq x) = o(1)l(x)/x$ as $x \to \infty$,

$$\sum_{j=1}^{n} E|X_j|I(|X_j| > \tau_j) = o(1)\sum_{j=1}^{n}\frac{l(\tau_j)}{\tau_j} \leq o(1)l(\tau_n)\sum_{j=1}^{n}\frac{1}{\tau_j}.$$

Since $1/\tau_n$ is a regularly varying function with index $-1/2$, by Tauberian theorem (see, for instance, Theorem 5 in Feller (1971), page 447), we have $\sum_{j=1}^{n}\frac{1}{\tau_j} \sim 2n/\tau_n$ as $n \to \infty$. Hence, as $n \to \infty$,

$$\sum_{j=1}^{n} E|X_j|I(|X_j| > \tau_j) = o(1)\frac{nl(\tau_n)}{\tau_n} = o(1)B_n/(\log\log n).$$

Thus (2.8) is proved and the proof of Lemma (2.2) is complete. □

Lemma 2.3. *For all $t \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P\Big(a(n)\max_{1\leq k<n} S_{k,n}^*/B_{k,n} \leq t + b(n)\Big) = \exp(-e^{-t}), \tag{2.9}$$

and

$$\lim_{n\to\infty} P\Big(a(n)\max_{1\leq k<n} |S_{k,n}^*|/B_{k,n} \leq t + b(n)\Big) = \exp(-2e^{-t}). \tag{2.10}$$

Proof. We only prove (2.9), since the proof of (2.10) is similar. Since $l(x^2) \leq 2^{C_0}l(x)$, by (42) in Csörgő *et al.* (2003), there exist two independent Wiener processes $W^{(1)}$ and $W^{(2)}$ such that, as $n \to \infty$,

$$S_n^* - W^{(1)}(B_n^2) = o(B_n/\sqrt{\log\log n}) \quad a.s. \tag{2.11}$$

and

$$\tilde{S}_n^* - W^{(2)}(B_n^2) = o(B_n/\sqrt{\log\log n}) \quad a.s. \tag{2.12}$$

Define $K_n = \exp\{\log^{1/3} n\}$ and

$$W(n,t) = \begin{cases} n^{-1/2}(W^{(1)}(nt) - t(W^{(1)}(n/2) + W^{(2)}(n/2))), & 0 \le t \le 1/2, \\ n^{-1/2}(-W^{(2)}(n-nt) + (1-t)(W^{(1)}(n/2) + W^{(2)}(n/2))), & 1/2 < t \le 1. \end{cases}$$

Computing its covariance function, one concludes that $W(n,t)$ is a Brownian bridge in $0 \le t \le 1$ for each $n \ge 1$. Now, as $n \to \infty$, we have

$$\sqrt{\log\log n} \max_{K_n \le k \le n/2} \Big| \frac{S^*_{k,n}}{B_{k,n}} - \frac{B_n^2 W(B_n^2, B_k^2/B_n^2)}{\sqrt{B_k^2(B_n^2 - B_k^2)}} \Big| \xrightarrow{P} 0. \tag{2.13}$$

To prove (2.13), we notice that for $k \le n/2$,

$$S^*_{k,n} = \frac{n}{k(n-k)} \Big(S^*_k - \frac{k}{n}(\tilde{S}^*_{n-[n/2]} + S^*_{[n/2]}) \Big).$$

Hence, for $k \le n/2$,

$$\begin{aligned} \Big| \frac{S^*_{k,n}}{B_{k,n}} - \frac{B_n^2 W(B_n^2, B_k^2/B_n^2)}{\sqrt{B_k^2(B_n^2 - B_k^2)}} \Big| &\le |W(B_n^2, B_k^2/B_n^2)| \Big| \frac{nB_n}{k(n-k)B_{k,n}} - \frac{B_n^2}{\sqrt{B_k^2(B_n^2 - B_k^2)}} \Big| \\ &\quad + \frac{nB_n}{k(n-k)B_{k,n}} \Big| \frac{k(n-k)}{nB_n} S^*_{k,n} - W(B_n^2, B_k^2/B_n^2) \Big| \\ &:= L_1(k,n) + L_2(k,n). \end{aligned} \tag{2.14}$$

First, we estimate $L_1(k,n)$. We have

$$\frac{k^2(n-k)^2 B^2_{k,n}}{n^2 B_n^2} - \frac{B_k^2(B_n^2 - B_k^2)}{B_n^4} = \Big(\frac{B_k^2}{B_n^2} - \frac{k}{n} \Big)^2 - \frac{k^2(B_n^2 - B^2_{[n/2]} - B^2_{n-[n/2]})}{n^2 B_n^2}.$$

Note that $(k/n)^{5/8} \le B_k/B_n \le (k/n)^{3/8}$ holds for all $K_n \le k \le n$ and sufficiently large n by the fact that B_n is a regularly varying function with index $1/2$. Then

$$\begin{aligned} \max_{K_n \le k \le n/(\log\log n)^5} \frac{B_n^3}{B_k^3} \Big(\frac{B_k^2}{B_n^2} - \frac{k}{n} \Big)^2 &\le 2 \max_{K_n \le k \le n/(\log\log n)^5} \Big(\frac{B_k}{B_n} + \frac{B_n^3 k^2}{B_k^3 n^2} \Big) \\ &\le 4(\log\log n)^{-5/8}. \end{aligned}$$

Also, by Lemma 2.1,

$$\begin{aligned} \max_{n/(\log\log n)^5 < k \le n/2} \frac{B_n^3}{B_k^3} \Big(\frac{B_k^2}{B_n^2} - \frac{k}{n} \Big)^2 &\le \max_{n/(\log\log n)^5 < k \le n/2} \frac{k^2 B_n^3}{n^2 B_k^3} \frac{(l(\eta_n) - l(\eta_{n/(\log\log n)^5}))^2}{l(\eta_n)^2} \\ &= o(1/\sqrt{\log\log n}), \quad n \to \infty. \end{aligned}$$

Hence, as $n \to \infty$,

$$\sqrt{\log\log n} \max_{K_n \le k \le n/2} \frac{B_n^3}{B_k^3} \Big(\frac{B_k^2}{B_n^2} - \frac{k}{n} \Big)^2 \to 0. \tag{2.15}$$

Again by Lemma 2.1,

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{B_n^3}{B_k^3}\frac{k^2(B_n^2-B_{[n/2]}^2-B_{n-[n/2]}^2)}{n^2B_n^2}\le\sqrt{\log\log n}\frac{l(\eta_n)-l(\eta_{[n/2]})}{l(\eta_n)}\to 0,$$

as $n\to\infty$. Thus,

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{B_n^3}{B_k^3}\Big|\frac{k^2(n-k)^2B_{k,n}^2}{n^2B_n^2}-\frac{B_k^2(B_n^2-B_k^2)}{B_n^4}\Big|\to 0,\quad n\to\infty. \tag{2.16}$$

This implies that for large n and all $K_n\le k\le n/2$,

$$\begin{aligned}\Big|\frac{k^2(n-k)^2B_{k,n}^2}{n^2B_n^2}-\frac{B_k^2(B_n^2-B_k^2)}{B_n^4}\Big| &\le \frac{1}{4}\frac{B_k^3}{B_n^3}\le\frac{1}{4}\frac{B_k^2(B_n^2-B_k^2)}{B_n^4}\frac{B_nB_{[n/2]}}{B_n^2-B_{[n/2]}^2}\\ &\le \frac{1}{2}\frac{B_k^2(B_n^2-B_k^2)}{B_n^4}.\end{aligned}$$

Hence, for large n and all $K_n\le k\le n/2$,

$$\frac{(1/2)B_n^2}{\sqrt{B_k^2(B_n^2-B_k^2)}}\le\frac{nB_n}{k(n-k)B_{k,n}}\le\frac{2B_n^2}{\sqrt{B_k^2(B_n^2-B_k^2)}}. \tag{2.17}$$

Noting that $|1/\sqrt{x}-1/\sqrt{y}|\le|x-y|/(x\sqrt{y})$ for all $x,y>0$, it follows from (2.16) and (2.17) that

$$\begin{aligned}&\sqrt{\log\log n}\max_{K_n\le k\le n/2}\Big|\frac{nB_n}{k(n-k)B_{k,n}}-\frac{B_n^2}{\sqrt{B_k^2(B_n^2-B_k^2)}}\Big|\\ \le\ &\sqrt{2\log\log n}\max_{K_n\le k\le n/2}\Big|\frac{k^2(n-k)^2B_{k,n}^2}{n^2B_n^2}-\frac{B_k^2(B_n^2-B_k^2)}{B_n^4}\Big|\Big(\frac{B_k^2(B_n^2-B_k^2)}{B_n^4}\Big)^{-3/2}\\ \le\ &4\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{B_n^3}{B_k^3}\Big|\frac{k^2(n-k)^2B_{k,n}^2}{n^2B_n^2}-\frac{B_k^2(B_n^2-B_k^2)}{B_n^4}\Big|\to 0.\end{aligned} \tag{2.18}$$

By properties of Brownian motion,

$$\begin{aligned}\max_{K_n\le k\le n/2}|W(B_n^2,B_k^2/B_n^2)| &\le 2B_n^{-1}\sup_{0\le t\le B_n^2}|W^{(1)}(t)|+B_n^{-1}|W^{(2)}(B_n^2/2)|\\ &\stackrel{d}{=} 2\sup_{0\le t\le 1}|W^{(1)}(t)|+|W^{(2)}(1/2)|.\end{aligned}$$

This together with (2.18) yields

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}L_1(k,n)\stackrel{P}{\to}0,\quad n\to\infty. \tag{2.19}$$

Next, we estimate $L_2(k,n)$. By (2.11) and (2.12),

$$\Big|\frac{k(n-k)}{nB_n}S_{k,n}^*-W(B_n^2,B_k^2/B_n^2)\Big|\le\frac{k}{nB_n}|W^{(1)}(B_n^2/2)-W^{(1)}(B_{[n/2]}^2)|$$

$$+\frac{k}{nB_n}|W^{(2)}(B_n^2/2)-W^{(2)}(B_{n-[n/2]}^2)|$$
$$+\Big|\frac{k}{n}-\frac{B_k^2}{B_n^2}\Big|\frac{|W^{(1)}(B_n^2/2)|+|W^{(2)}(B_n^2/2)|}{B_n}+\frac{o_k(1)B_k}{B_n\sqrt{\log\log k}},$$

where $o_k(1)\to 0$ as $k\to\infty$. Similarly to the proof of (2.15), we have

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{B_n}{B_k}\Big|\frac{B_k^2}{B_n^2}-\frac{k}{n}\Big|\to 0,\quad n\to\infty.$$

This, together with (2.17) and the fact that

$$\frac{|W^{(1)}(B_n^2/2)|+|W^{(2)}(B_n^2/2)|}{B_n}\overset{d}{=}|W^{(1)}(1/2)|+|W^{(2)}(1/2)|,$$

as $n\to\infty$, yields

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{nB_n}{k(n-k)B_{k,n}}\Big|\frac{k}{n}-\frac{B_k^2}{B_n^2}\Big|\frac{|W^{(1)}(B_n^2/2)|+|W^{(2)}(B_n^2/2)|}{B_n}\xrightarrow{P}0.$$

Similarly to the proof of Lemma 2.1, we have

$$\frac{\sqrt{\log\log n}}{B_n}|W^{(1)}(B_n^2/2)-W^{(1)}(B_{[n/2]}^2)|\overset{d}{=}\sqrt{\log\log n}\Big(\frac{B_n^2/2-B_{[n/2]}^2}{B_n^2}\Big)^{1/2}|W^{(1)}(1)|$$
$$=\sqrt{\log\log n}\Big(\frac{(n/2)l(\eta_n)-[n/2]l(\eta_{[n/2]})}{nl(\eta_n)}\Big)^{1/2}|W^{(1)}(1)|\xrightarrow{P}0,\quad n\to\infty.$$

Hence, by (2.17), as $n\to\infty$,

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{nB_n}{k(n-k)B_{k,n}}\frac{k}{nB_n}|W^{(1)}(B_n^2/2)-W^{(1)}(B_{[n/2]}^2)|\xrightarrow{P}0.$$

Similarly, as $n\to\infty$,

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{nB_n}{k(n-k)B_{k,n}}\frac{k}{nB_n}|W^{(2)}(B_n^2/2)-W^{(2)}(B_{n-[n/2]}^2)|\xrightarrow{P}0.$$

Also, by (2.17), as $n\to\infty$,

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}\frac{nB_n}{k(n-k)B_{k,n}}\frac{o_k(1)B_k}{B_n\sqrt{\log\log k}}\xrightarrow{P}0.$$

Hence

$$\sqrt{\log\log n}\max_{K_n\le k\le n/2}L_2(k,n)\xrightarrow{P}0,\quad n\to\infty. \tag{2.20}$$

Now (2.13) follows from (2.14), (2.19) and (2.20). Now, similarly, as $n\to\infty$,

$$\sqrt{\log\log n}\max_{n/2<k\le n-K_n}\Big|\frac{S_{k,n}^*}{B_{k,n}}-\frac{B_n^2W(B_n^2,B_k^2/B_n^2)}{\sqrt{B_k^2(B_n^2-B_k^2)}}\Big|\xrightarrow{P}0.$$

Hence, as $n \to \infty$,

$$\sqrt{\log\log n}\Big|\max_{K_n\le k\le n-K_n}\frac{S^*_{k,n}}{B_{k,n}} - \sup_{K_n\le k\le n-K_n}\frac{W(B_n^2,t)}{\sqrt{(B_k^2/B_n^2)(1-B_k^2/B_n^2)}}\Big| \xrightarrow{P} 0.$$

Next, we will show that, as $n \to \infty$,

$$\sqrt{\log\log n}\Big|\sup_{\frac{B^2_{K_n}}{B_n^2}\le t\le\frac{B^2_{n-K_n}}{B_n^2}}\frac{W(B_n^2,t)}{\sqrt{t(1-t)}} - \sup_{K_n\le k\le n-K_n}\frac{W(B_n^2,t)}{\sqrt{(B_k^2/B_n^2)(1-B_k^2/B_n^2)}}\Big| \xrightarrow{P} 0. \quad (2.21)$$

Write

$$\Delta_n = \inf_{K_n+1\le k\le n-K_n}\frac{B_k^2-B_{k-1}^2}{B_n^2} = \frac{l(\eta_{K_n})}{B_n^2}$$

and recall that $W(B_n^2,t)$ is a Brownian bridge in $t\in[0,1]$ for each $n\ge 1$. Hence, to prove (2.21), we only need to show that, as $n\to\infty$,

$$\sqrt{\log\log n}\sup_{\frac{B^2_{K_n}}{B_n^2}\le t,s\le\frac{B^2_{n-K_n}}{B_n^2}}\ \sup_{|t-s|\le\Delta_n}\Big|\frac{W(t)-tW(1)}{\sqrt{t(1-t)}} - \frac{W(s)-sW(1)}{\sqrt{s(1-s)}}\Big| \xrightarrow{P} 0,$$

where $W(t)$ is a standard Brownian motion. This follows from results on the increments of a Brownian motion (see for instance Csörgő and Révész (1981), Theorem 1.2.1) and by some basic calculations. We omit the details here. Hence, as $n\to\infty$,

$$\sqrt{\log\log n}\Big|\max_{K_n\le k\le n-K_n}\frac{S^*_{k,n}}{B_{k,n}} - \sup_{\frac{B^2_{K_n}}{B_n^2}\le t\le\frac{B^2_{n-K_n}}{B_n^2}}\frac{W(B_n^2,t)}{\sqrt{t(1-t)}}\Big| \xrightarrow{P} 0. \quad (2.22)$$

By using (A.4.30) and (A.4.31) in Csörgő and Horváth (1997), as $n\to\infty$, we conclude

$$(2\log\log B_n^2)^{-1/2}\sup_{1/B_n^2\le t\le c(B_n^2)}\frac{W(B_n^2,t)}{\sqrt{t(1-t)}} \xrightarrow{P} \sqrt{5/12},$$

$$(2\log\log B_n^2)^{-1/2}\sup_{1-c(B_n^2)\le t\le 1/B_n^2}\frac{W(B_n^2,t)}{\sqrt{t(1-t)}} \xrightarrow{P} \sqrt{5/12},$$

where $c(B_n^2) = \exp\{(\log B_n^2)^{5/12}\}/B_n^2$. Notice that $B^2_{K_n}/B_n^2 \le c(B_n^2)$ and $B^2_{n-K_n}/B_n^2 \ge 1-c(B_n^2)$ for sufficiently large n. Hence, as $n\to\infty$,

$$a(B_n^2)\sup_{1/B_n^2\le t\le B^2_{K_n}/B_n^2}\frac{W(B_n^2,t)}{\sqrt{t(1-t)}} - b(B_n^2) \xrightarrow{P} -\infty, \quad (2.23)$$

$$a(B_n^2)\sup_{B^2_{n-K_n}/B_n^2\le t\le 1-1/B_n^2}\frac{W(B_n^2,t)}{\sqrt{t(1-t)}} - b(B_n^2) \xrightarrow{P} -\infty. \quad (2.24)$$

By (A.4.29) and Theorem A.3.1 in Csörgő and Horváth (1997), we arrive at

$$\lim_{n\to\infty} P\Big(a(B_n^2) \sup_{1/B_n^2\le t\le 1-1/B_n^2} \frac{W(B_n^2,t)}{\sqrt{t(1-t)}} \le t + b(B_n^2)\Big) = \exp(-e^{-t}). \tag{2.25}$$

Now, from (2.22)–(2.25) it follows that for all $t\in\mathbb{R}$,

$$\lim_{n\to\infty} P\Big(a(B_n^2) \max_{K_n\le k\le n-K_n} S_{k,n}^*/B_{k,n} \le t + b(B_n^2)\Big) = \exp(-e^{-t}).$$

This, together with (2.28) below, implies that for all $t\in\mathbb{R}$,

$$\lim_{n\to\infty} P\Big(a(B_n^2) \max_{1\le k<n} S_{k,n}^*/B_{k,n} \le t + b(B_n^2)\Big) = \exp(-e^{-t}).$$

Since, as $n\to\infty$, $\log\log B_n^2 = \log\log n + o(1)$, we have

$$\begin{aligned}
&a(n) \max_{1\le k<n} S_{k,n}^*/B_{k,n} - b(n)\\
&\quad= \frac{a(n)}{a(B_n^2)}\Big(a(B_n^2)\max_{1\le k<n} S_{k,n}^*/B_{k,n} - b(B_n^2)\Big) + \frac{a(n)}{a(B_n^2)} b(B_n^2) - b(n)\\
&\quad= (1+o(1))\Big(a(B_n^2)\max_{1\le k<n} S_{k,n}^*/B_{k,n} - b(B_n^2)\Big) + o(1),
\end{aligned}$$

which implies (2.9). Lemma 2.3 is proved. □

Proof of Theorem 1.2. Write $K_n = \exp\{\log^{1/3} n\}$, and put

$$\Omega_1 = \Big\{K_n < k \le n/4 : \sum_{i=1}^k |Z_i| \le B_k/\log\log k\Big\},$$

$$\Omega_2 = \Big\{K_n < k \le n/4 : \sum_{i=1}^k |\tilde{Z}_i| \le B_k/\log\log k\Big\}.$$

Define $\Omega' = \Omega_1 \cup \{k : n/4 < k \le n/2\}$, $\Omega'' = \{k : n-k \in \Omega_2\} \cup \{k : n/2 < k < 3n/4\}$ and $\Omega_1' = \{k : 2 \le k \le n/4\} - \Omega_1$, $\Omega_2' = \{k : 3n/4 \le k \le n-2\} - \{k : n-k\in\Omega_2\}$.

Notice that, as $n\to\infty$, $S_{[nt]}/b_n \xrightarrow{d} W(t)$ and $V_n^2/b_n^2 \xrightarrow{P} 1$, where W is a Brownian motion and b_n is a regularly varying function with index 1/2. Hence

$$\begin{aligned}
&\frac{\min_{k\le n/4}(\tilde{V}_{n-[n/2]}^2 + V_{[n/2]}^2 - V_k^2 - (\tilde{S}_{n-[n/2]} + S_{[n/2]} - S_k)^2/(n-k))}{b_n^2}\\
&\quad\ge \frac{\tilde{V}_{n-[n/2]}^2}{b_n^2} - \frac{3\tilde{S}_{n-[n/2]}^2 + 6(\max_{1\le k\le n/2}|S_k|)^2}{(n/2)b_n^2} \xrightarrow{P} 1/2, \quad n\to\infty.
\end{aligned}$$

Notice that by the self-normalized LIL of Griffin and Kuelbs (1989), as $n\to\infty$, we have

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2\log\log n(V_n^2 - S_n^2/n)}} = 1 \quad a.s.$$

Consequently,

$$\frac{1}{\sqrt{2\log\log n}} \max_{2\le k\le K_n} \frac{|S_k|}{\sqrt{(V_k^2 - S_k^2/k)}} \le \frac{\sqrt{2\log\log K_n}}{\sqrt{2\log\log n}}(1+o(1)) = \sqrt{1/3} + o(1) \quad a.s.$$

Similarly, by (18) in Csörgő *et al.* (2003), we conclude

$$\frac{1}{\sqrt{2\log\log n}} \max_{k>K_n \text{ and } k\in\Omega_1'} \frac{|S_k|}{\sqrt{(V_k^2 - S_k^2/k)}} \le \sqrt{1/2} + o(1) \quad a.s., \quad n\to\infty.$$

Thus, by noting that $\frac{a+b}{\sqrt{c+d}} \le \frac{a}{\sqrt{c}} + \frac{b}{\sqrt{d}}$ holds for all $a, b, c, d > 0$,

$$\begin{aligned}\frac{1}{\sqrt{2\log\log n}} \max_{k\in\Omega_1'} \frac{|S_{k,n}|}{\bar{V}_{k,n}} &\le \frac{1}{\sqrt{2\log\log n}} \max_{k\in\Omega_1'} \frac{n}{n-k} \frac{|S_k|}{\sqrt{V_k^2 - S_k^2/k}} \\ &+ \frac{(|S_{[n/2]}| + |\tilde{S}_{n-[n/2]}|)/(b_n\sqrt{2\log\log n})}{\min_{k\le n/4} \sqrt{\tilde{V}_{n-[n/2]}^2 + V_{[n/2]}^2 - V_k^2 - (\tilde{S}_{n-[n/2]} + S_{[n/2]} - S_k)^2/(n-k)}/b_n} \\ &\le 2\sqrt{2}/3 + o_P(1), \quad n\to\infty.\end{aligned}$$

This, as $n\to\infty$, implies

$$a(n) \max_{k\in\Omega_1'} \frac{|S_{k,n}|}{\bar{V}_{k,n}} - b(n) \xrightarrow{P} -\infty, \tag{2.26}$$

and, similarly

$$a(n) \max_{k\in\Omega_2'} \frac{|S_{k,n}|}{\bar{V}_{k,n}} - b(n) \xrightarrow{P} -\infty. \tag{2.27}$$

Furthermore, similarly, by using (20) in Csörgő *et al.* (2003), and by the facts that, as $n\to\infty$, $S_n^*/B_n \xrightarrow{d} N(0,1)$ and $\limsup_{n\to\infty} S_n^*/(2B_n^2\log\log n)^{1/2} = 1$ *a.s.* (by (2.11)), we infer

$$a(n) \max_{k\in\Omega_1'\cup\Omega_2'} \frac{|S_{k,n}^*|}{B_{k,n}} - b(n) \xrightarrow{P} -\infty. \tag{2.28}$$

Now, in order to prove Theorem 1.2, we only need to show that, as $n\to\infty$,

$$a(n) \max_{k\in\Omega'} \Big| \frac{S_{k,n}}{V_{k,n}} - \frac{S_{k,n}^*}{B_{k,n}} \Big| \xrightarrow{P} 0, \tag{2.29}$$

and

$$a(n) \max_{k\in\Omega''} \Big| \frac{S_{k,n}}{V_{k,n}} - \frac{S_{k,n}^*}{B_{k,n}} \Big| \xrightarrow{P} 0. \tag{2.30}$$

In fact, if (2.29) and (2.30) hold true, then it follows from (2.28) and Lemma 2.3 that, for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} P\Big(a(n) \max_{k\in\Omega'\cup\Omega''} S_{k,n}/V_{k,n} \le t + b(n)\Big) = \exp(-e^{-t}). \tag{2.31}$$

And also by Lemma 2.3, we obtain that

$$\frac{1}{\sqrt{2\log\log n}} \max_{1\le k<n} \frac{|S^*_{k,n}|}{B_{k,n}} \xrightarrow{P} 1, \qquad n\to\infty. \tag{2.32}$$

By noting that

$$V^2_{k,n} \le \bar{V}^2_{k,n} \le \max\Big\{\frac{k}{k-1}, \frac{n-k}{n-k-1}\Big\} V^2_{k,n},$$

and by applying (2.29), (2.30) and (2.32), we get that,

$$\begin{aligned} a(n) \max_{k\in\Omega'\cup\Omega''} \Big|\frac{S_{k,n}}{\bar{V}_{k,n}} - \frac{S_{k,n}}{V_{k,n}}\Big| &\le \frac{a(n)}{\sqrt{K_n}} \max_{k\in\Omega'\cup\Omega''} \frac{|S_{k,n}|}{V_{k,n}} \\ &\le \frac{a(n)}{\sqrt{K_n}} \max_{k\in\Omega'\cup\Omega''} \Big|\frac{S_{k,n}}{V_{k,n}} - \frac{S^*_{k,n}}{B_{k,n}}\Big| + \frac{a(n)}{\sqrt{K_n}} \max_{k\in\Omega'\cup\Omega''} \frac{|S^*_{k,n}|}{B_{k,n}} \\ &\xrightarrow{P} 0, \quad n\to\infty. \end{aligned} \tag{2.33}$$

This, together with (2.26), (2.27) and (2.31), yields Theorem 1.2.

Now we go to prove (2.29) and (2.30). We only prove (2.29), since the proof of (2.30) is similar. Clearly, we have

$$\begin{aligned} a(n)\max_{k\in\Omega'}\Big|\frac{S_{k,n}}{V_{k,n}} - \frac{S^*_{k,n}}{B_{k,n}}\Big| &\le a(n)\max_{k\in\Omega'}\Big|\frac{S_{k,n}}{V_{k,n}} - \frac{S_{k,n}}{B_{k,n}}\Big| + a(n)\max_{k\in\Omega'}\Big|\frac{S_{k,n}-S^*_{k,n}}{B_{k,n}}\Big| \\ &\le a(n)\max_{k\in\Omega'}\Big|\frac{S_{k,n}}{V_{k,n}}\frac{V^2_{k,n}-B^2_{k,n}}{B^2_{k,n}}\Big| + a(n)\max_{k\in\Omega'}\Big|\frac{S_{k,n}-S^*_{k,n}}{B_{k,n}}\Big|. \end{aligned} \tag{2.34}$$

By the self-normalized LIL of Griffin and Kuelbs (1989), we get that, as $n\to\infty$,

$$\sup_{K_n\le k\le n/2} \frac{V^2_{k,n}}{V^2_k/k^2 + (\tilde{V}^2_{n-[n/2]} + V^2_{[n/2]} - V^2_k)/(n-k)^2} \to 1 \quad a.s.$$

Hence, for sufficiently large n,

$$\begin{aligned} a(n)\max_{k\in\Omega'}\Big|\frac{S_{k,n}}{V_{k,n}}\frac{V^2_{k,n}-B^2_{k,n}}{B^2_{k,n}}\Big| \quad &\le \quad 2a(n)\max_{k\in\Omega'}\Big|\frac{S_k}{V_k}\frac{V^2_{k,n}-B^2_{k,n}}{B^2_{k,n}}\Big| \\ &\quad +2a(n)\frac{V_n}{\tilde{V}_{n-[n/2]}}\max_{k\in\Omega'}\Big|\frac{S_{[n/2]}-S_k}{V_n}\frac{V^2_{k,n}-B^2_{k,n}}{B^2_{k,n}}\Big| \\ &\quad +2a(n)\max_{k\in\Omega'}\frac{|\tilde{S}_{n-[n/2]}|}{\tilde{V}_{n-[n/2]}}\Big|\frac{V^2_{k,n}-B^2_{k,n}}{B^2_{k,n}}\Big|. \end{aligned} \tag{2.35}$$

Since $EX = 0$ and $E|X_1|^r < \infty$ for any $1 < r < 2$, it follows from the Marcinkiewicz-Zygmund strong law of large number (c.f. Chow and Teicher (1978), page 125) that $S_n/n^{1/r} \to 0$ $a.s.$ Hence, as $n \to \infty$,

$$\frac{(\log\log n)S_n^2}{nB_n^2} \to 0 \quad a.s.$$

Note that for $n/4 < k \le n/2$,

$$\frac{\sum_{j=1}^{k}(Z_j^2 + |EZ_j|^2)/k^2}{B_{[n/2]}^2/(n-k)^2} \le 9\frac{\sum_{j=1}^{k}(Z_j^2 + |EZ_j|^2)}{B_{[n/2]}^2},$$

and, by Lemma 2.2,

$$\begin{aligned}
&\frac{\sum_{j=1}^{n}|Z_j|^2}{B_n^2/\log\log n} \le \Big(\frac{\sum_{j=1}^{n}|Z_j|}{B_n/\sqrt{\log\log n}}\Big)^2 \xrightarrow{P} 0,\\
&\frac{\sum_{j=1}^{n}|EY_j|^2}{B_n^2/\log\log n} = \frac{\sum_{j=1}^{n}|EZ_j|^2}{B_n^2/\log\log n} \le \Big(\frac{\sum_{j=1}^{n}|EZ_j|}{B_n/\sqrt{\log\log n}}\Big)^2 \to 0, \quad n \to \infty.
\end{aligned}$$

Now, by (40) of Csörgő *et al.* (2003), we have

$$\begin{aligned}
(\log\log n)\max_{k\in\Omega'}\Big|\frac{V_{k,n}^2 - B_{k,n}^2}{B_{k,n}^2}\Big| &\le 3\max_{k\in\Omega'}\frac{\log\log k|\sum_{j=1}^{k}(Y_j^2 - EY_j^2)|}{B_k^2}\\
&+\frac{\log\log n|\sum_{j=1}^{[n/2]}(Y_j^2 - EY_j^2)|}{B_{[n/2]}^2} + \frac{\log\log n|\sum_{j=1}^{n-[n/2]}(\tilde{Y}_j^2 - EY_j^2)|}{B_{n-[n/2]}^2}\\
&+3\max_{k\in\Omega_1}\frac{\log\log k\sum_{j=1}^{k}(Z_j^2 + |EY_j|^2)}{B_k^2} + 10\frac{\log\log n\sum_{j=1}^{[n/2]}(Z_j^2 + |EY_j|^2)}{B_{[n/2]}^2}\\
&+\frac{\log\log n\sum_{j=1}^{n-[n/2]}(\tilde{Z}_j^2 + |EY_j|^2)}{B_{n-[n/2]}^2} + 12\max_{k\in\Omega'}\frac{(\log\log k)S_k^2}{kB_k^2}\\
&+3\frac{(\log\log n)\tilde{S}_{n-[n/2]}^2}{(n/2)B_{n-[n/2]}^2} + 3\frac{(\log\log n)S_{[n/2]}^2}{(n/2)B_{[n/2]}^2} \xrightarrow{P} 0, \quad n \to \infty. \qquad (2.36)
\end{aligned}$$

By the self-normalized LIL of Griffin and Kuelbs (1989), we conclude

$$\max_{k\le n/2}\frac{|S_{[n/2]} - S_k|}{V_n\sqrt{2\log\log n}} \le \frac{2\max_{k\le n/2}|S_k|}{V_n\sqrt{2\log\log n}} \le 2 \quad a.s, \quad n \to \infty. \qquad (2.37)$$

By the facts that $V_n^2/b_n^2 \xrightarrow{P} 1$ and $\tilde{V}_n^2/b_n^2 \xrightarrow{P} 1$, as $n \to \infty$, we get

$$\frac{V_n}{\tilde{V}_{n-[n/2]}} = \frac{V_n}{b_n^2}\frac{b_{n-[n/2]}^2}{\tilde{V}_{n-[n/2]}}\frac{b_n^2}{b_{n-[n/2]}^2} \xrightarrow{P} 2. \qquad (2.38)$$

Thus, by using (2.35)-(2.38) and applying again the self-normalized LIL of Griffin and Kuelbs (1989), as $n \to \infty$, we arrive at

$$a(n) \max_{k \in \Omega'} \left| \frac{S_{k,n}}{V_{k,n}} \frac{V_{k,n}^2 - B_{k,n}^2}{B_{k,n}^2} \right| \xrightarrow{P} 0. \tag{2.39}$$

Similarly to the proof of (2.36), by using Lemma 2.2, we have

$$\begin{aligned} a(n) \max_{k \in \Omega'} \left| \frac{S_{k,n} - S_{k,n}^*}{B_{k,n}} \right| &\leq \sqrt{3} \max_{k \in \Omega_1} \frac{\sqrt{\log\log k}\sum_{j=1}^{k}(|Z_j| + |EZ_j|)}{B_k} \\ &+ 4 \frac{\sqrt{\log\log n}\sum_{j=1}^{[n/2]}(|Z_j| + |EZ_j|)}{B_{[n/2]}} + \frac{\sqrt{\log\log n}\sum_{j=1}^{n-[n/2]}(|\tilde{Z}_j| + |EZ_j|)}{B_{n-[n/2]}} \\ &\xrightarrow{P} 0, \quad n \to \infty. \end{aligned} \tag{2.40}$$

Now (2.29) follows from (2.34), (2.39) and (2.40). This also completes the proof of Theorem 1.2. □

References

[1] Brodsky, B. E. and Darkhovsky, B. S. (1993). *Nonparametric Methods in Change-Point Problems.* Kluwer, Dordrecht.

[2] Chow, Y. S. and Teicher, H.(1978).*Probability Theory* . Springer-Verlag, New York.

[3] Csörgő, M. and Horváth, L. (1988). Nonparametric methods for changepoint problems. In *Quality Control and Reliability* (P.R. Krishnaiah, C.R., Rao, eds.), Handbook of Statistics **7**, Elsevier, Amsterdam, 403C425.

[4] Csörgő, M. and Horváth, L. (1997) *Limit Theorems in Change-Point Analysis.* Wiley, New York.

[5] Csörgő, M. and Révész P. (1981). *Strong Approximations in Probability and Statistics.* Probability and Mathematical Statistics. Academic Press.

[6] Csörgő, M., Szyszkowicz, B. and Wang, Q. (2003). Darling–Erdős theorem for self-normalized sums. *Ann. Probab.* **31**, 676-692.

[7] Csörgő, M., Szyszkowicz, B. and Wang, Q. (2004). On Weighted Approximations and Strong Limit Theorems for Self-normalized Partial Sums Processes. In *Asymptotic Methods in Stochastics*, (L. Horváth, B. Szyszkowicz, eds.), 489–521, Fields Inst. Commun. **44**, Amer. Math. Soc., Providence RI.

[8] Csörgő, M., Szyszkowicz, B. and Wang, Q. (2008a). On weighted approximations in D[0,1] with applications to self-normalized partial sum processes. *Acta Mathematica Hungarica* **121**, 307-332.

[9] Csörgő, M., Szyszkowicz, B. and Wang, Q. (2008b). Asymptotics of Studentized U-type processes for changepoint problems. *Acta Mathematica Hungarica* **121**, 333-357.

[10] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications.* Vol. 2, Wiley, New York.

[11] Galambos, J. and Seneta, E. (1973). Regularly Varying Sequences. *Proc. Amer. Math. Soc.* **41**, 110-116.

[12] Gombay, E. and Horváth, L. (1994). An application of the maximum likelihood test to the changepoint problem. *Stochastic Processes and their Applications* **50**, 161-171.

[13] Gombay, E. and Horváth, L. (1996a). Applications for the time and change and the power function in change-point models. *Journal of Statistical Planning and Inference* **52**,: 43C66.

[14] Gombay, E. and Horváth, L. (1996b). On the rate of approximations for maximum likelihood test in change-point models. *Journal of Multivariate Analysis* **56**, 120C152.

[15] Griffin, P. and Kuelbs, J. (1989). Self-normalized laws of the iterated logarithm. *Ann. Probab.* **17**, 1571-1601.

[16] Orasch, M. and Pouliot, W. (2004). Tabulating weighted sup-norm functionals used in change-point analysis. *Journal of Statistical Computation and Simulation* **74**, 249-276.